Killing Forms, W-Invariants, and the Tensor Product Map

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- S. Garibaldi, A. Merkurjev, J.-P. Serre construct $Q(G)$ in [1].
- $Q(G)$ Appears in work by S. Garibaldi [2], S. Baek [3], as well as by A. Merkurjev, A. Neshitov, and K. Zainoulline [4] relating to cohomological invariants of linear algebraic groups.

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- In particular W acts on the symmetric tensor product $\mathrm{S}(T^*)$, and so we can discuss invariant quadratic forms.
- $\mathrm{S}^2(T^*)^W = \mathrm{Q}(G)$.

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- Analagous to the Killing form in Lie theory, $\mathcal{K}(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ on $\mathrm{Lie}(G)$.

W-Invariants

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- If G is semisimple, $\mathrm{S}^2(T^*)^W = \mathbb{Z}\langle q_1\rangle \oplus \ldots \oplus \mathbb{Z}\langle q_m\rangle$.

Examples

Group	Killing Form	Normalized Killing Form
$SL(V)$, $\dim(V) = n+1$	$4(n+1)\sum_{\substack{i,j=1 \\ i\leq j}}^{n} e_i e_j$	$\sum_{\substack{i,j=1 \\ i\leq j}}^{n} e_i e_j$
$SO(V)$, $\dim(V) = 2n$	$4(n-1)\sum_{i=1}^{n} e_i^2$	$\sum_{i=1}^{n} e_i^2$
$SO(V)$, $\dim(V) = 2n+1$	$4(n-2)\sum_{i=1}^{n} e_i^2$	$\sum_{i=1}^{n} e_i^2$
$Sp(V)$, $\dim(V) = 2n$	$4(n+1)\sum_{i=1}^{n} e_i^2$	$\sum_{i=1}^{n} e_i^2$

Where in call cases $T^* = \langle e_i \mid 1 \leq i \leq n \rangle$.

Induced Map on W-Invariants

- Q(G) is functorial. If $\rho\colon G \to H$ is a homomorphism we have

$$\rho^*\colon \mathrm{S}^2(T_H^*)^W \to \mathrm{S}^2(T_G^*)^W$$

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- Since $\mathrm{S}^2(T_H^*)^W$ is generated by some normalized Killing forms $q_1, \ldots, q_m$, this map is described by their images, called the Rost multpliers of ρ.

Kroenecker Tensor Product Map

- The tensor product map

$$\begin{aligned}\rho\colon \mathrm{GL}(V_1)\times \mathrm{GL}(V_2) &\to \mathrm{GL}(V_1\otimes V_2)\\ (A,B) &\mapsto A\otimes B\end{aligned}$$

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$$(A, B) \mapsto A \otimes B$$

- If $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$, $A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}$.

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- In general we consider

$$\rho\colon \mathrm{GL}(V_1) \times \dots \times \mathrm{GL}(V_m) \to \mathrm{GL}(V_1 \otimes \dots \otimes V_m)$$
$$(A_1, \dots, A_m) \mapsto A_1 \otimes \dots \otimes A_m$$

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- For the following cases
  - $G_1, \ldots, G_n, H = \mathrm{SL}$
  - $G_1, \ldots, G_{2m} = \mathrm{Sp}$
    $G_{2m+1}, \ldots, G_n = \mathrm{SO}$
    $H = \mathrm{SO}$
  - $G_1, \ldots, G_{2m+1} = \mathrm{Sp}$
    $G_{2m+2}, \ldots, G_n = \mathrm{SO}$
    $H = \mathrm{Sp}$

  we consider

$$\rho \colon G_1(V_1) \times \ldots \times G_n(V_n) \to H(V_1 \otimes \ldots \otimes V_n)$$

Example Computation of ρ^*

$\rho\colon \mathsf{SO}(\mathbb{F}^{2n+1}) \times \mathsf{SO}(\mathbb{F}^{2m+1}) \to \mathsf{SO}(\mathbb{F}^{(2n+1)(2m+1)})$

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- Choose $T_{2n+1} = \{\mathrm{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$ and others similarily.

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- $T_{2n+1}^* = \langle e_i \mid 1 \leq i \leq n \rangle$ where $e_i(\mathrm{diag}(t_1, \ldots, t_1^{-1})) = t_i$. $T_{(2n+1)(2m+1)}^* = \langle f_i \mid 1 \leq i \leq 2nm + n + m \rangle$

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- Choose $T_{2n+1} = \{\mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$ and others similarily.
- $T^*_{2n+1} = \langle e_i \mid 1 \le i \le n \rangle$ where $e_i(\mathrm{diag}(t_1, \dots, t_1^{-1})) = t_i$. $T^*_{(2n+1)(2m+1)} = \langle f_i \mid 1 \le i \le 2nm + n + m \rangle$
-

$$\rho^*(f_i) = \begin{cases} & 0 \le k \le n-1 \\ (e_{k+1}, e_r) & \quad 1 \le r \le m \\ (e_{k+1}, 0) & \quad r = m+1 \\ (e_{k+1}, -e_{2m+2-r}) & \quad m+2 \le r \le 2m+1 \\ & k = n \\ (0, e_r) & \quad 1 \le r \le m \end{cases}$$

where $i = k(2m+1) + r$ with $0 \le k \le 2n$ and $1 \le r \le 2m+1$.

Example Computation of ρ^*

- $\rho^*(q_{(2n+1)(2m+1)}) = \rho^*\left(\sum_{i=1}^{2nm+n+m} f_i^2\right) = \sum_{i=1}^{2nm+n+m} \rho^*(f_i)^2.$

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- $\vdots$
- $((2m+1)q_{2n+1}, (2n+1)q_{2m+1}).$

Results

Theorem

Let $V_1, \dots, V_n$ be vector spaces such that $\dim(V_i) = d_i$. Consider linear algebraic groups $G_1, \dots, G_n, H$ in one of the previous configurations (where $G_i = \mathrm{Sp}$ only when d_i is even).
Consider the Kronecker product map

$$\rho\colon G_1(V_1) \times \ldots \times G_n(V_n) \to H(V_1 \otimes \ldots \otimes V_n)$$

and let $q_1, \dots, q_n, q_H$ be the respective normalized Killing forms. Then

$$\rho|_n^*(q_H) =$$
$$\Big((d_2 \ldots d_n)q_1, \dots, (d_1 \ldots \hat{d}_i \ldots d_n)q_i, \dots, (d_1 \ldots d_{n-1})q_n\Big)$$

where $\hat{d}_i$ represents ommision.

Thank You.

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- S. Garibaldi. *Orthogonal Involutions on Algebras of Degree 16 and the Killing Form of E_8*. Contemporary Mathematics Volume 493, 2009.
- S. Baek. *Chow Groups of Products of Severi-Brauer Varieties and Invariants of Degree 3*. arXiv:1502.03023v2, 2015.
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